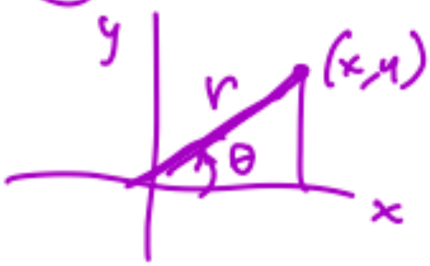
We will go over/check 3.2ab, so keep your hmk til the end.

From last time:

<u>Example</u>. Find $\iint (xy-3x)\, dA$ over the region that is inside the unit circle.

Change to polar coordinates!



$x = r\cos\theta$
$y = r\sin\theta$

$$dA = dx\,dy = dx \wedge dy$$

wedge product of differential forms.

$$dx = (\cos\theta)dr - r\sin\theta\, d\theta$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

Algebra rules for wedge product.

$$da \wedge db = -db \wedge da$$

$$da \wedge da = 0$$

imagine an arrow sticking out according to the right hand rule.

$$dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$$

dx dy dz

$$dx = (\cos\theta)dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dx \wedge dy = (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$

usual distributive property - be careful with order.

$$= \cos\theta\, dr \wedge \sin\theta\, dr + \cos\theta\, dr \wedge r\cos\theta\, d\theta$$
$$- r\sin\theta\, d\theta \wedge \sin\theta\, dr - r\sin\theta\, d\theta \wedge r\cos\theta\, d\theta$$

$$= \cos\theta\sin\theta\, \underset{0}{dr\wedge dr} + r\cos^2\theta\, dr\wedge d\theta$$

$$- r\sin^2\theta\, \underbrace{d\theta\wedge dr}_{-dr\wedge d\theta} - r^2\sin\theta\cos\theta\, \underset{0}{d\theta\wedge d\theta}$$

$$= (r\cos^2\theta + r\sin^2\theta)\, dr\wedge d\theta$$

$$= r(\cos^2\theta + \sin^2\theta)\, dr\wedge d\theta$$

$$= r\, dr\wedge d\theta$$

$$\boxed{dx\wedge dy = r\, dr\wedge d\theta \text{ in polar coordinates}}$$

$$I = \iint (xy - 3x)\, \underset{dx\wedge dy}{dA}$$

over the region that is inside the unit circle.

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left[(r\cos\theta)(r\sin\theta) - 3(r\cos\theta)\right] r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(r^3 \cos\theta \sin\theta - 3r^2 \cos\theta \right) dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{r^4}{4} \cos\theta \sin\theta - r^3 \cos\theta \right) \Big|_0^1 d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{1}{4} \cos\theta \sin\theta - \cos\theta \right) d\theta$$

$u = \sin\theta \quad du = \cos\theta\, d\theta$

$$= \underbrace{\int_0^0 \frac{1}{4} u\, du}_{0} + - \int_0^{2\pi} \cos\theta\, d\theta$$

$$-\sin\theta \Big|_0^{2\pi}$$

$$= \boxed{0}.$$

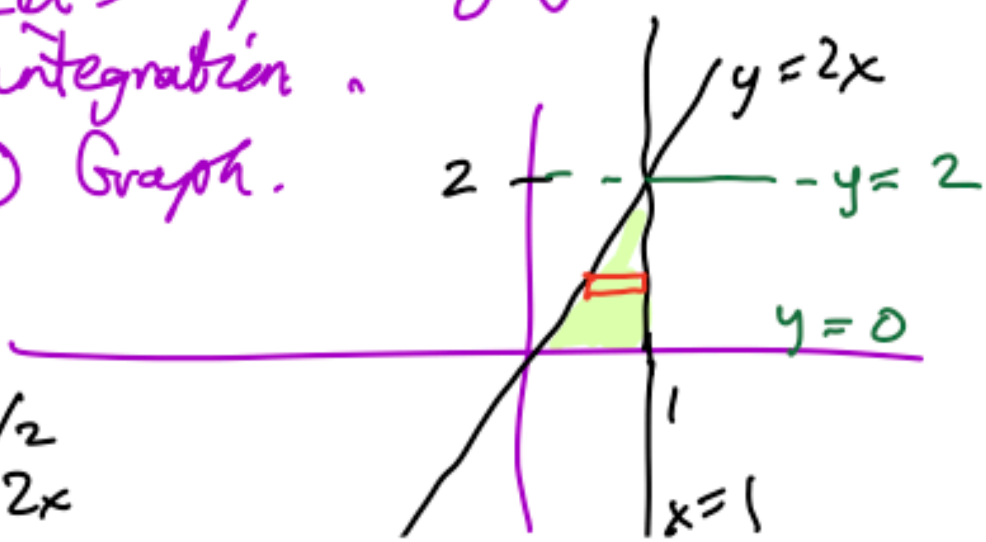
Note: we could tell this is zero, because $xy - 3x$ is positive & negative the same amount on different sides of the unit circle, so the volumes would cancel.

Example: Find $\displaystyle\int_{y=0}^{y=2}\int_{x=y/2}^{x=1} e^{x^2}\,dx\,dy.$

Problem: there is no known antideriv of e^{x^2}.

Let's try changing the order of integration.

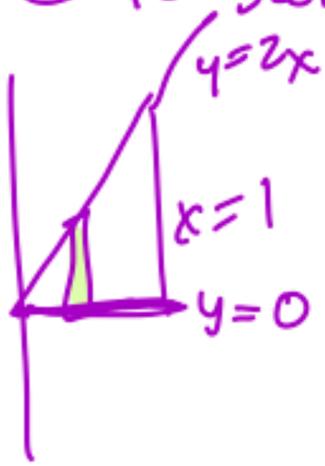
① Graph.



$x = y/2$

$y = 2x$

(2) To switch, use vertical slices:



$$\int_{x=0}^{1}\left(\int_{y=0}^{y=2x} e^{x^2}\,dy\right)dx$$

$$= \int_{x=0}^{1}\left(y e^{x^2}\Big|_{y=0}^{2x}\right)dx$$

$$= \int_{x=0}^{1} 2x e^{x^2}\,dx$$

Let $u = x^2$

$du = 2x\,dx$

$$= \int_{u=0}^{1} e^{u}\,du = e^{u}\Big|_0^1 = \boxed{e-1}.$$

Homework exercises:

3.2 a
$$\int_0^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2 \, dy \, dx$$

① evaluate
② graph
③ switch order

$$= \int_{x=0}^{2} \left(\frac{y^3}{3} \Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \right) dx$$

$$= \int_{x=0}^{2} \frac{2(4-x^2)^{3/2}}{3} \, dx$$

Let $x = 2\sin u$ $\qquad x=0 \leftrightarrow u=0$
$\qquad x=2 \leftrightarrow u=\frac{\pi}{2}$

$dx = 2\cos u \, du$

$$= \int_{u=0}^{u=\pi/2} \frac{2}{3}\left(4-4\sin^2 u\right)^{3/2} 2\cos u \, du$$

$$\left(4\left(1-\sin^2 u\right)\right)^{3/2}$$

$$= \qquad \underset{8}{4^{3/2}} \left(\cos^2(u)\right)^{3/2}$$

$$= \frac{32}{3}\int_{u=0}^{\pi/2} \cos^4 u \, du$$

$$\cos^4 u = \left(\cos^2 u\right)^2$$

$$\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos(2u)$$

$$\cos^4 u = \frac{1}{4}\left(1+\cos(2u)\right)^2$$

$$= \frac{1}{4}\left(1 + 2\cos(2u) + \underline{\cos^2(2u)}\right)$$

$$= \frac{1}{4}\left(1 + 2\cos(2u) + \frac{1}{2} + \frac{1}{2}\cos(4u)\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + 2\cos(2u) + \frac{1}{2}\cos(4u)\right)$$

$$= \frac{3}{8} + \frac{1}{2}\cos(2u) + \frac{1}{8}\cos(4u)$$

$$\text{integral} = \frac{32}{3}\int_0^{\pi/2}\left(\frac{3}{8} + \frac{1}{2}\cos(2u) + \frac{1}{8}\cos(4u)\right)du$$

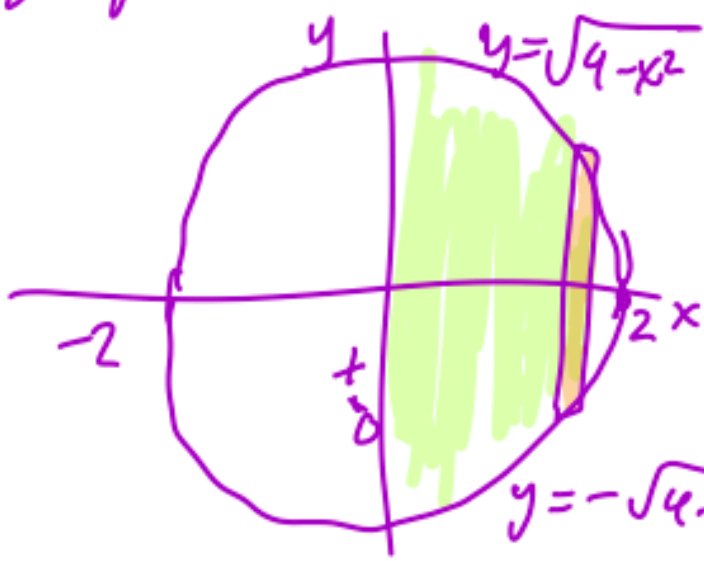
$$= \frac{32}{3}\left[\frac{3}{8}u + \frac{1}{4}\sin(2u) + \frac{1}{32}\sin(4u)\right]\Bigg|_0^{\pi/2}$$

$$= \frac{32}{3}\cdot\left[\frac{3}{8}\frac{\pi}{2} + 0 + 0\right]$$

$$= \boxed{2\pi}.$$

$$\int_0^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2\,dy\,dx$$

graph



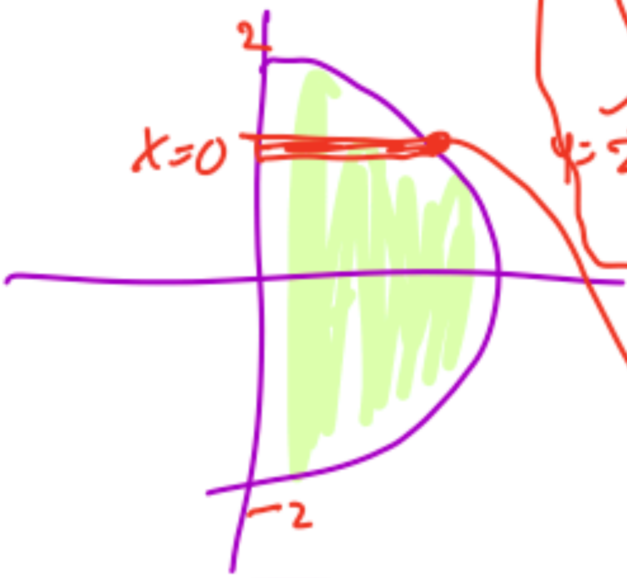
$y = -\sqrt{4-x^2}$

$y^2 = 4-x^2$

$x^2+y^2=4$

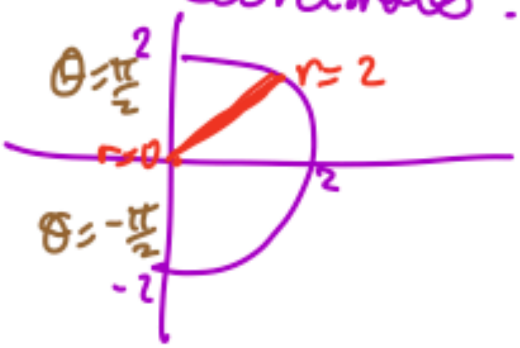
Horiz slices

$$\left(\int_{y=-2}^{2}\left(\int_{x=0}^{\sqrt{4-y^2}} y^2\,dx\right)dy\right)$$



$x=\sqrt{4-y^2}$

How would we do this using polar coordinates?



$$dx\wedge dy = r\,dr\wedge d\theta$$

$$x=r\cos\theta$$

$$y=r\sin\theta$$

$$y^2=r^2\sin^2\theta$$

$$\int_{\theta=-\frac{\pi}{2}}^{\pi/2}\int_{r=0}^{2} r^2\sin^2\theta\, r\,dr\,d\theta$$

r^3

-1

$$\int_{\theta=-\pi/2}^{\pi/2} \left(\frac{r^4}{4}\sin^2\theta \,\Big|_{r=0}^{r=2} \right) d\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} 4\sin^2\theta \, d\theta$$

$$4\left(\underbrace{\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta)}_{\sin^2\theta}\right)$$

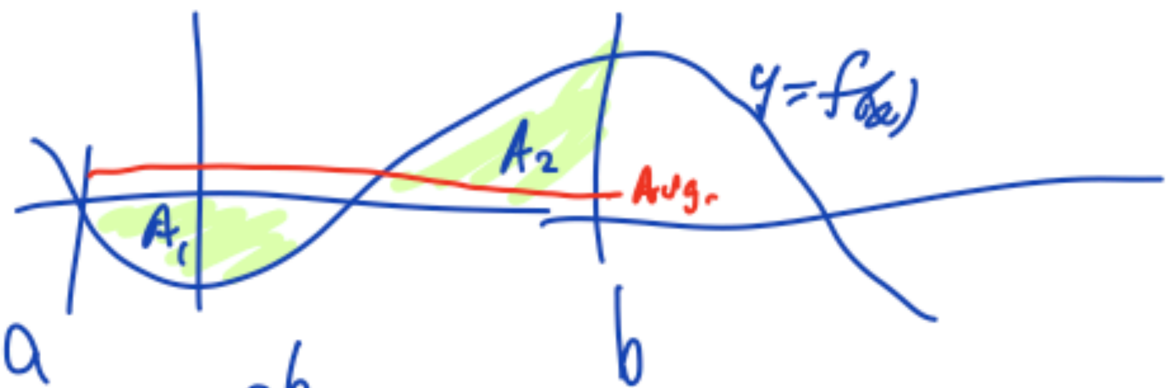
$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \left(2 - 2\cos(2\theta)\right) d\theta$$

$$= 2\theta - \sin(2\theta) \Big|_{-\pi/2}^{\pi/2}$$

$$= \left(2\left(\tfrac{\pi}{2}\right) - 0\right) - \left(2\left(\tfrac{-\pi}{2}\right) - 0\right)$$

$$= \pi + \pi = \boxed{2\pi}.$$

Interpretations of Integrals.



$$\int_a^b f(x)\,dx = A_2 - A_1 .$$

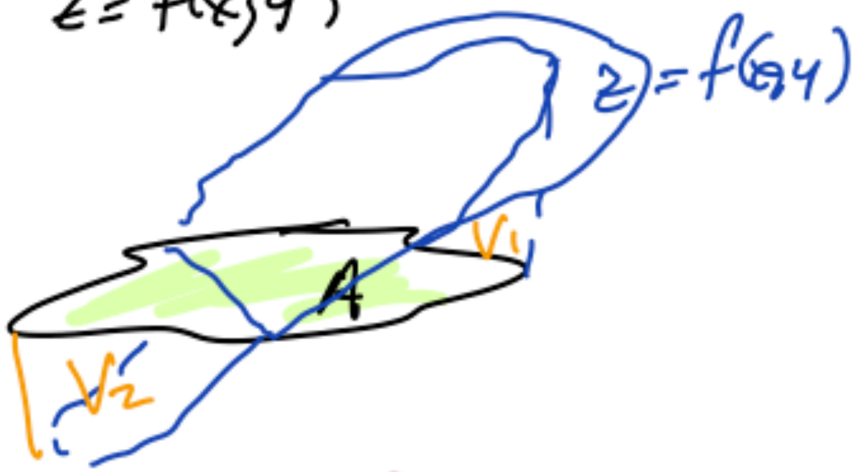
$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\int_a^b f(x)\,dx}{\int_a^b 1\,dx}$$

= average value of $f(x)$
for $a \le x \le b$

2-d:

$$\iint_A f(x,y)\,dx\,dy = V_1 - V_2$$

$z = f(x,y)$



$$\frac{1}{\text{area}(A)} \iint_A f(x,y)\,dx\,dy = \frac{\iint_A f(x,y)\,dx\,dy}{\iint_A 1\,dx\,dy}$$

= average value of $f(x,y)$ for $(x,y) \in A$.

Notice $\iint_A 1\,dx\,dy = A \cdot 1$

$= A$

$$\iiint_V 1 \; dx, dy, dz = \text{volume}(V)$$

Triple integral

$$\iiint_R f(x,y,z) \; dx, dy, dz$$

= limit of sum $f(x,y,z) \cdot \underbrace{(\Delta x \Delta y \Delta z)}_{\text{box volume}}$ over R.

$$\frac{1}{\text{Volume}(R)} \iiint_R f(x,y,z)\, dx\, dy\, dz$$

$=$ average value of $f(x,y,z)$ over (x,y,z) in R.

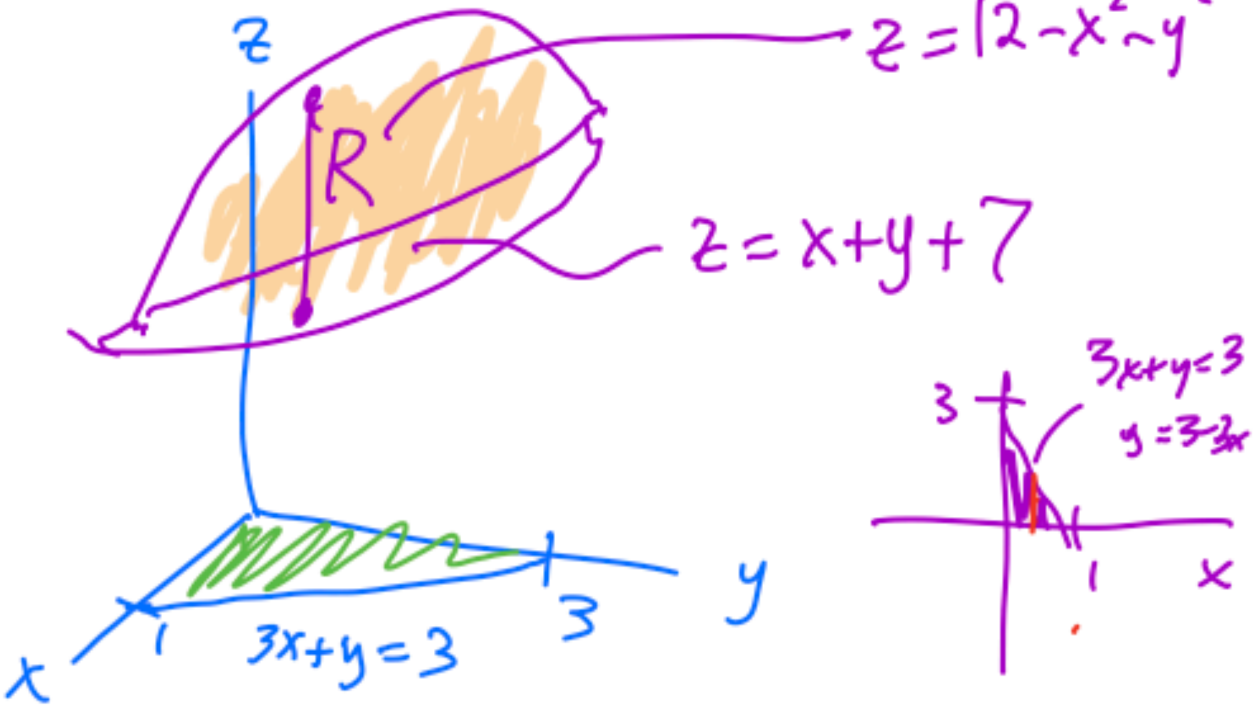
$$= \frac{\iiint_R f(x,y,z)\, dx\, dy\, dz}{\iiint_R 1\, dx\, dy\, dz}.$$

$$\Rightarrow \iiint_R f(x,y,z)\, dx\, dy\, dz$$

$$= \begin{pmatrix} \text{avg value} \\ \text{of } f(x,y,z) \\ \text{over } R \end{pmatrix} \cdot \begin{pmatrix} \text{Volume} \\ \text{of} \\ R \end{pmatrix}$$

Setting up triple integrals

(1 typical method)



$$\iiint_R (x+2yz)\,dx\,dy\,dz = ?$$

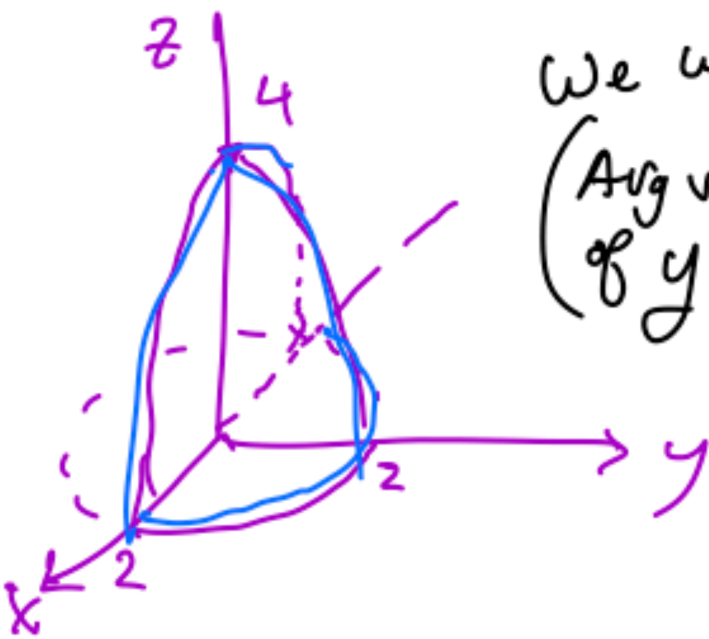
upper surface

$$= \int_{x=0}^{1}\int_{y=0}^{3-3x}\int_{z=x+y+7}^{z=12-x^2-y^2} (x+2yz)\,dz\,dy\,dx$$

lower surface

inside integral bounds can depend on the outside variables.

<u>Example</u>: Let Ω be the region between the surface $z = 4 - x^2 - y^2$, the xy plane, and the xz plane in the part where $z > 0$, $y > 0$. Find the average value of y in this region.



We want

$$\begin{pmatrix} \text{Avg value} \\ \text{of } y \end{pmatrix} = \frac{\iiint_\Omega y \, dx\,dy\,dz}{\iiint_\Omega 1 \, dx\,dy\,dz}$$